Advanced Probabilistic Structural Analysis Method for Implicit Performance Functions

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In probabilistic structural analysis, the performance or response functions usually are implicitly defined and must be solved by numerical analysis methods such as finite-element methods. In such cases, the commonly used probabilistic analysis tool is the mean-based second-moment method, which provides only the first two statistical moments. This paper presents an advanced mean-based method, which is capable of establishing the full probability distributions to provide additional information for reliability design. The method requires slightly more computations than the mean-based second-moment method but is highly efficient relative to the other alternative methods. Several examples are presented to demonstrate the method. In particular, the examples show that the new mean-based method can be used to solve problems involving nonmonotonic functions that result in truncated distributions.

Nomenclature

F	= cumulative distribution function
f	= probability density function
f_n	= natural frequency
H	= higher-order terms
n	= number of random variables
$P[\cdot]$	= probability of
p_f	= probability of failure
u	= standard normal variable
X	= design random variable
x	= real number associated with X
Z	= response function or performance function
Z_1	= linear approximation of Z
z,z_0	= real number associated with Z
μ	= mean value
σ	= standard deviation
ν	= Poisson's ratio
Φ	= standard normal cumulative distribution function

Superscript

$*$	= most probable point

Introduction

The need to address the uncertainties in a design has long been recognized. Traditionally, designers use safety factors to provide confidence. However, the safety factor approach is questionable because it usually does not take into account the underlying probability distributions. For example, consider the $\mu \pm 3\sigma$ design approach. Assuming that the underlying distribution is Gaussian, then the probability of exceeding the three standard deviation bounds is 0.0027. In practice, however, non-Gaussian distributions are not uncommon, and the true probabilities cannot be determined without the knowledge of the underlying distributions. For consistent reliability design, therefore, it is important to determine the probability distributions. Unfortunately, the evaluations of the probability distributions for structural performance functions generally involve complicated numerical computations that prohibit the use of the standard Monte Carlo simulation method.

Because of the preceding problem, a commonly used method in probabilistic structural analysis is the mean-based second-moment method, which is used to obtain the mean and the standard deviation of the unknown distribution.[1,2] However, without the knowledge of higher moments, the first two statistical moments can only provide a probability bound, which is, according to Chebyshev's inequality,

$$P\{|X-\mu| \geq K\sigma\} \leq \frac{1}{K^2} \tag{1}$$

where K is a real number. This probability bound is known to be generally too conservative. For example, for $K = 3$, the probability bound is 0.11, which is far greater than the value 0.0027 for a normally distributed random variable. In general, more than two moments will be required to "define" a distribution.

Recently, numerical methods have been proposed to compute higher statistical moments and to establish the cumulative distribution function (CDF) by curve fitting.[3,4] These efforts recognize the need for defining the full distribution. However, these numerical moment methods generally must compute the performance function (e.g., the structural response) at well-designed calculation points (e.g., quadrature points) and tend to be inefficient as the number of random variables is large. A more effective procedure is therefore needed to establish the CDF, one that can be used for implicit performance functions where the performance or response of the system to changes in random variables is not explicit.

A probabilistic structural analysis computer program, NESSUS, is being developed as part of a Probabilistic Structural Analysis Methods (PSAM) program funded by NASA.[5,6] To compute the structural response CDF effectively, a new probabilistic analysis tool called the advanced mean-value, first-order method[7] (AMVFO) has been developed. The major feature of this method is its capability to approximate the CDF with few extra computations, relative to the mean-based second-moment method. In this paper, since only the first-order method is presented, the AMVFO method will be abbreviated as the AMV method.

This paper summarizes the previous AMV method and its limitations to certain nonlinear performance functions and

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presents a generalized AMV method that is applicable to more general nonlinear performance functions. In this paper, the generalized AMV method will be applied to several examples to demonstrate the effectiveness of the method in generating the CDFs for strongly nonmonotonic performance functions.

Review of Probabilistic Structural Analysis Methods

In structural reliability analysis, a "performance function" or "limit state function" $g(X)$ is often formulated in terms of a vector of basic design factors $X = (X_1, X_2, \ldots, X_n)$ in which X_i are random variables. In this paper, it is assumed that X is a vector of mutually independent variables. Dependent random variables can be treated using transformations.[7,8,11]

The limit state which separates the design space into "failure" and "safe" regions is $g(X) = 0$. The probability of failure is

$$p_f = P[g < 0] \tag{2}$$

An exact solution of p_f requires the integration of a multiple integral denoted as

$$p_f = \int_{\Omega} f_x(x)\, \mathrm{d}x \tag{3}$$

where $f_x(x)$ is the joint probability density function of X, and Ω is the failure region. The solution of this multiple integral is, in general, extremely complicated. Alternatively, a Monte Carlo solution provides a convenient but usually time-consuming approximation. For practical purposes, efficient, approximate analysis tools are needed. Current approaches, based on analytical approximation methods, include a fast probability integration algorithm[9] and the first- and second-order reliability methods.[8]

For each limit state, the *most probable point* is defined as follows. First, transform the nonnormal variables X to standard, normal variables u using the following transformation:

$$x = F_X^{-1}[\Phi(u)] \tag{4}$$

On the limit-state surface, $g(u) = 0$, the most probable point is the point that defines the minimum distance from the origin, $u = 0$ to the limit-state surface. This point can be found by an optimization scheme or other iteration algorithms such as the Rackwitz-Fiessler algorithm.[8,10]

Define $Z(X)$ as a structural performance function; the CDF of $Z(X)$ can then be formulated as

$$P(Z < z_0) = F_Z(z_0) = \int_{Z < z_0} f_x(x)\, \mathrm{d}x \tag{5}$$

By varying z_0, a series of limit states can be formed. A most-probable-point-locus can be defined by connecting the most probable points.

In theory, it is possible to repeatedly apply the previous structural reliability methods to compute the CDF. In reality, however, many problems may occur. The problems include inefficiency and convergence instability due to search algorithms and multiple minimum distance points. Thus, there is a need to develop a more efficient and robust procedure to establish the CDF.

Mean-Based Methods

Assume that the Z function is "smooth" or can be smoothed, and Taylor's series expansion of Z exists at the mean values. The Z function can be expressed as

$$\begin{aligned} Z(X) &= Z(\mu) + \sum_{i=1}^{n} \left(\frac{\partial Z}{\partial X_i}\right) \cdot (X_i - \mu_i) + H(X) \\ &= a_0 + \sum_{i=1}^{n} a_i X_i + H(X) \\ &= Z_1(X) + H(X) \end{aligned} \tag{6}$$

where the derivatives are evaluated at the mean values; Z_1 is a random variable representing the sum of the first-order terms, and $H(X)$ represents the higher-order terms. The mean-based methods are defined as the methods based on the mean values expansion.

Mean-Value First-Order Method

By retaining only the first-order terms, the mean and the standard deviation of Z are:

$$\mu_Z \approx a_0 + \sum_{i=1}^{n} a_i \mu_{X_i} \tag{7}$$

$$\sigma_Z^2 \approx \sum_{i=1}^{n} a_i^2 \sigma_{X_i}^2 \tag{8}$$

These two statistical moments can be computed easily after the coefficients a_i are computed. In structural analysis, there are several ways of obtaining a_i including the direct differentiation method and the adjoint method.[12] The NESSUS code utilizes iterative perturbation algorithms[13] to efficiently compute the Z functions around an x and uses the solutions to estimate a_i. In general, the coefficients a_i can be computed by numerical differentiation and the minimum required number of Z-function evaluations is $(n + 1)$.

Equations (7) and (8) constitute the mean-based, first-order, second-moment method. However, when the probability distributions, not just the first two moments, of X_i are fully defined, the CDF of the first-order terms Z_1 is also fully defined. Since the Z_1 function is linear and explicit, its CDF can be computed effectively using many methods, including the structural reliability analysis methods mentioned earlier. Thus, the mean-value first-order (MVFO, abbreviated as MV) solution defines the CDF of Z_1, not just the two moments.

For nonlinear Z functions, the MV solution is, in general, not sufficiently accurate. For simple problems, it is possible to use higher-order expansions to improve the accuracy. For example, a mean-value second-order (MVSO) solution can be obtained by retaining second-order terms in the series expansion. However, for problems involving implicit Z functions and large n, the higher-order approach becomes difficult and inefficient. The AMV method described below provides an alternative to improve the MV solution with minimum additional Z-function evaluations.

Advanced Mean-Value First-Order Method[7,14]

The AMV method takes the MV solution one step further to compute the CDF. The key to the AMV method is the reduction of the truncation error by replacing the higher-order terms $H(X)$ by a simplified function $H(Z_1)$ dependent on Z_1. Ideally, the $H(Z_1)$ function should be based on the exact most-probable-point locus (MPPL) of the Z function to optimize the truncation error. The AMV procedure simplifies this procedure by using the MPPL of Z_1. As a result of this approximation, the truncation error is not optimum; however, because the Z-function correction points are generally "close" to the exact most probable points, the AMV solution provides reasonably good CDF estimations.

The stepwise AMV (first-order) procedure can be summarized as follows:

1) Obtain the Z_1 function based on perturbations about the mean values.

2) Compute the CDF of Z_1 at selected z_0 points using the fast probability integration method (other approximation techniques such as the first- and second-order reliability methods can also be used).

3) Select a number of CDF values that cover a sufficiently wide probability range.

4) For each CDF value, identify the most probable point x^*.

5) Recompute $Z(x^*)$ to replace z_0 for the same CDF in the previous step.

The preceding steps require the construction of the Z_1 function only once for all the CDF levels. Assuming that a numerical differentiation scheme is used to define the Z_1 function, the required number of the Z function evaluations is $(n+1+m)$, where n is the number of random variables and m the number of CDF levels.

Iteration Algorithms

The accuracy of the preceding AMV method can be further improved by using the exact MPPL to define the H function. Based on the AMV results, two iteration algorithms, one for specified probability level and the other for specified Z level have been proposed[7] to improve the CDF estimates. For completeness, the suggested algorithms for specified probability level, which will be used in the demonstration examples, is summarized in the following steps:

1) Construct the Z_1 function, initially mean-based, and search for z_0 such that $P[Z_1 < z_0] =$ probability goal.

2) Use most probable point of $Z_1 = z_0$ and recompute Z.

3) Obtain the new Z_1 function around the most probable point of $Z_1 = z_0$.

4) Repeat the above steps unitl z_0 converges.

In general, the above steps require the construction of the Z_1 function several times. Therefore, for complicated Z functions which require extensive computations, efficient sensitivity computation schemes are preferred in updating the Z_1 function.

To illustrate the AMV (first-order) method and the iteration algorithm, consider an example where the Z function is the first bending natural frequency of a cantilever beam. This frequency can be approximated as

$$f_n = 0.5602\sqrt{\frac{Et^2}{12\rho L^4}} \tag{9}$$

where E is the modulus of elasticity, t is the thickness, ρ is the material density, and L is the length.

By numerical differentiation, at the mean values of the random variables, the following first-order approximation can be obtained using the results of five Z-function [i.e., Eq. (9)] evaluations.

$$f_{n1} = a_0 + a_1 E + a_2 t + a_3 \rho + a_4 L \tag{10}$$

Based on Eq. (10), an MV CDF solution can be obtained as shown in Fig. 1. Note that the CDF solution is plotted on a normal probability paper.

The preceding MV solution is exact if f_n is a linear function of the four random variables. However, since Eq. (9) is a nonlinear function, Eq. (10) is subjected to error in the regions away from the mean values. For each selected probability, the AMV solution is obtained by calculating f_n at the most probable point (E^*, t^*, ρ^*, L^*) obtained using Eq. (10). This requires one function evaluation of f_n for each selected probability.

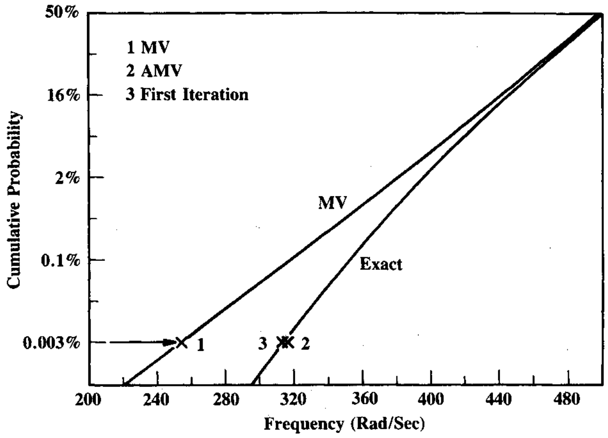

Fig. 1 AMV method and iteration procedure.

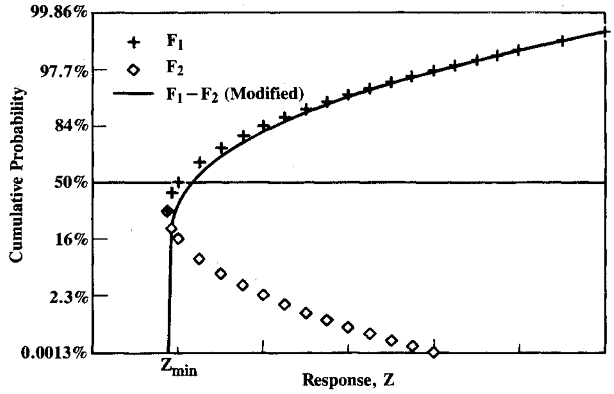

Fig. 2 Modified AMV solution for concave Z function.

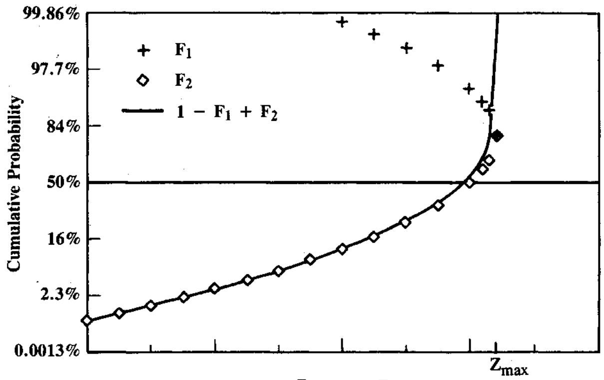

Fig. 3 Modified AMV solution for convex Z function.

In Fig. 1, three points are shown at a selected probability level (CDF = 0.003%). Point 1 is the MV solution, point 2 is the AMV solution, and point 3 is the AMV solution using the described iteration algorithm.

Generalized Advanced Mean-Value Method

The previous AMV (first-order) method and the associated iteration algorithms have been applied to a set of selected problems to validate the NESSUS code.[15] In those tested examples, the AMV method provided satisfactory solutions. In fact, the errors in the AMV probability solutions were often much less than the errors due to the finite-element modeling. However, recently it has been found that this AMV method has one weakness analogous to the "multiple minimum distance points" problem in structural reliability analysis. No general solution procedure was available to solve this problem except the Monte Carlo method. However, when the Z function is a strongly nonmonotonic function of x, the AMV-based CDF solution may suggest that the Z function is a strongly nonmonotonic function of Z_1. Based on this information, modified AMV solutions can be developed.

Based on its formulation, the previous AMV solution procedure inherently assumes one H value for each Z value. Thus, the method is suitable if there exists only one significant most probable point for a Z value. It may be suspected that the CDF result may be in great error if there is more than one Z_1 solution regions for some Z values. This could happen when the performance function is strongly nonmonotonic.

To alleviate the problem, a generalized AMV method can be formulated by assuming that Z is a function of random variable Z_1. Since the AMV solution provides the CDF of Z_1 (the MV solution) and also the relationship between Z and Z_1

(the AMV solution), the CDF of Z is uniquely defined. This formulation, of course, is approximate because the relationship between Z and Z_1 is only approximate. However, this approximation does not require any extra Z-function calculations and is able to provide the information on the characteristic of the Z function (e.g., monotonic or nonmonotonic) and to suggest a CDF modification procedure.

In general, a more accurate solution would require first the identification of all the significant most probable points and then the modification of the probability solution by assuming multiple-limit states. The system reliability analysis methods, particularly the reliability bounds theory,[16] may be applied to estimate the probability. For practical purposes, however, such an analysis procedure may be too complicated to use. In the following, a simple procedure for computing an approximate solution is proposed.

For the cases where the Z function is a concave- or a convex-like function of Z_1, a simple modification procedure based on the generalized AMV method can be summarized as follows: 1) For a concave function, the CDF plot would identify a minimum value of z, $z_{\min}$. For $z > z_{\min}$, two CDFs, F_1 and F_2, can be identified for each z. Let $F_1 > F_2$; the modified CDF, based on the theory of functions of one random variable, is simply $(F_1 - F_2)$. 2) For a convex function, a $z_{\max}$ can be identified. For $z < z_{\max}$, the modified CDF is $[1-(F_1-F_2)]$. The procedure to modify the AMV solution is illustrated in Figs. 2 and 3 for concave and convex Z functions, respectively. Note that these two figures represent the cases where there are two Z_1 solutions for some Z values; i.e., they are analogous to the two most-probable-point cases.

Applying the theory of functions of one random variable, it is straightforward to extend the procedure to the situations where there are more than two Z_1 solutions for a Z value.

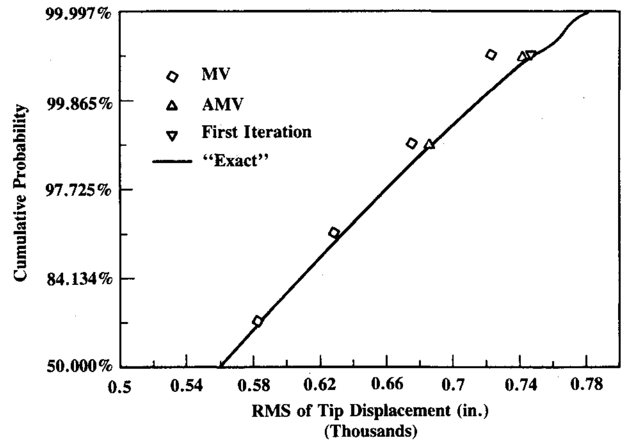

Fig. 4 NESSUS validation problem.

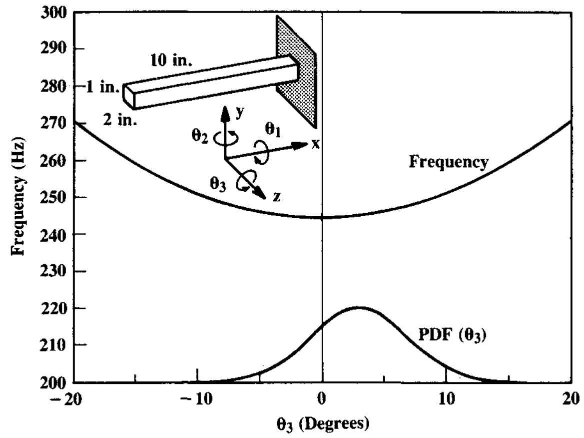

Fig. 5 Frequency as a function of material orientation angle.

Table 1 Variables for example 1

Variable	Mean	COV[a]	Distribution
E	6.895E + 4 MPa	0.03	Lognormal
L	0.508 m	0.01	Normal
t	0.0249 m	0.01	Normal
ξ	0.05	0.01	Normal
W_A	6.45E − 4 m^2/s^3-rad	0.10	Lognormal
ρ	22.26 kg/m^3	0.02	Normal

[a]Coefficient of variation.

Table 2 Variables for example 2

Variable	Mean	Standard deviation
Θ_1	−5 deg	3.87 deg
Θ_2	2 deg	3.87 deg
Θ_3	3 deg	3.87 deg
E_x	1.267E + 5 MPa	0
E_y	1.267E + 5 MPa	0
G	1.285E + 5 MPa	0
ν_x	0.386	0
ν_y	0.386	0
L	0.254 m	0
t	0.0254 m	0
ρ	716.9 kg/m^3	0

However, it should be noted that, when a function is extremely nonlinear (e.g., a sinusoidal wave shape), the number of CDF points required to fully describe the nonlinear shape may be high; therefore, care should be exercised to generate sufficient AMV solution points to accurately define the shape. When a highly nonlinear shape is detected, it may be necessary to perform a Monte Carlo study to supplement the AMV solution. A procedure that applies an importance sampling technique to confirm or improve the AMV solution is currently under development.

Numerical Examples

Three examples are selected to demonstrate the AMV method. The first example shows the typical AMV solution in which the CDF does not appear to have truncation limits. In this case, no modification to the AMV solution is needed. The remaining two examples involve strong nonmonotonic performance functions that can be solved using the generalized AMV method as just described.

Example 1: Random Vibration

This problem was developed to test the NESSUS random vibration capabilities. In this example, a cantilever beam is subjected to random base excitation. The random variables include the modulus of elasticity E, material density ρ, damping factor ξ, length L, thickness t, and constant acceleration power spectral density (PSD) level W_A. The PSD is modeled as a truncated white noise with cutoff frequency properly selected such that the random loading would excite, approximately, only the first mode. The random variables are defined in Table 1.

Using a single-degree-of-freedom model, the tip root-mean-square (rms) displacement can be approximated as

$$\text{rms} = \sqrt{\frac{1.707L^6 W_A \rho^{1.5}}{E^{1.5} t^3 \xi}} \tag{11}$$

The NESSUS probabilistic solution is shown in Fig. 4. In this figure, the MV solution employs the mean-based sensitivity analysis result to establish the approximate, linear Z_1 function [see Eq.(6)]; the AMV solution uses the information from the MV solution, and the recomputation of the response at the approximate most probable point. Note that no modification to the AMV solution is needed because the CDF curve is clearly monotonic.

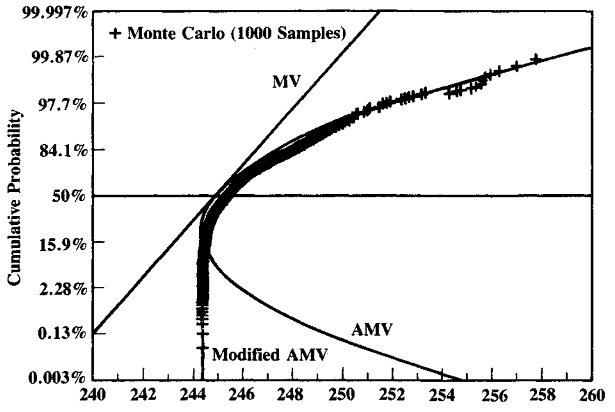

Fig. 6 Modified AMV solution for example 2 (case 1: one random angle).

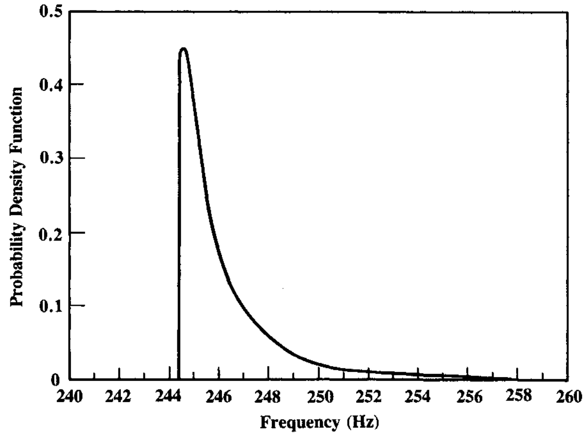

Fig. 7 Probability density function of the natural frequency.

To improve the result further, the iteration algorithms just discussed can be applied. A first iteration solution for a CDF value is presented in Fig. 4, which shows excellent agreement between the NESSUS solution and the "exact" solution based on the Monte Carlo simulation (100,000 samples). Note that there was a 3.9% difference between the NESSUS result and the analytical result at the 50% probability level. This difference has been used to adjust the exact solution.

Based on Fig. 4, it is obvious that the AMV solution improves the MV solution significantly; whereas the first iteration solution improves the AMV solution slightly. Since the AMV accuracy generally deteriorates at the extreme tail regions of the distribution, it is suggested that the iterations should be performed at the critical tail regions.

Example 2: Material Orientation Effect

A recent probabilistic analysis of a single crystal turbine blade has shown that the AMV solution of the first bending frequency CDF is truncated; i.e., the frequency has a lower bound. To study the truncation phenomena, a simple beam model, as shown in Fig. 5, was developed to simulate the turbine blade. The beam is made of a single crystal material, and the material orientation is defined by three angles. The data are shown in Table 2 in which E_x and E_y are the "off-axis" moduli.

Two cases are considered in this study. The first case considers that only one material orientation angle is random and is normally distributed. This allows a simple closed form equation for more detailed study. The second case considers three normally distributed random orientation angles, and the NESSUS code is applied to solve the problem.

Case 1: One Random Angle

The first-mode frequency is

$$f_n = 0.5602\sqrt{\frac{E_1 t^2}{12\rho L^4}} \tag{12}$$

where E_1 is the "on-axis" (or material symmetry axis) modulus, which can be shown[17] to be a function of the off-axis moduli and the random angle θ_3 as

$$E_1 = \frac{EG}{(c^4 + s^4 - 2c^2s^2\nu_y)G + c^2s^2E} \tag{13}$$

where $E = E_x = E_y$, $c = \cos(\theta_3)$, and $s = \sin(\theta_3)$. By substituting Eq. (13) into Eq. (12), the relationship between f_n and θ_3 is shown in Fig. 5, which shows that f_n is a nonmonotonic function in the probability significant region. Furthermore, the frequency has a minimum.

By numerical differentiation, the MV solution was obtained as

$$f_{n1} = 254.96 + 0.42396(\theta_3 - 3) \tag{14}$$

Based on Eq. (14), f_{n1} is normally distributed. Therefore, the CDF of f_{n1} is linear on a normal probability paper, as shown in Fig. 6. The AMV solution is shown in Fig. 6 and appears to be parabolic with a minimum of θ_3. The parabolic shape indicates that f_n is a nonmonotonic function of θ_3; therefore, a modification to the AMV solution is needed.

By applying the generalized AMV procedure, the modified CDF becomes truncated at the left tail. To verify the result, a

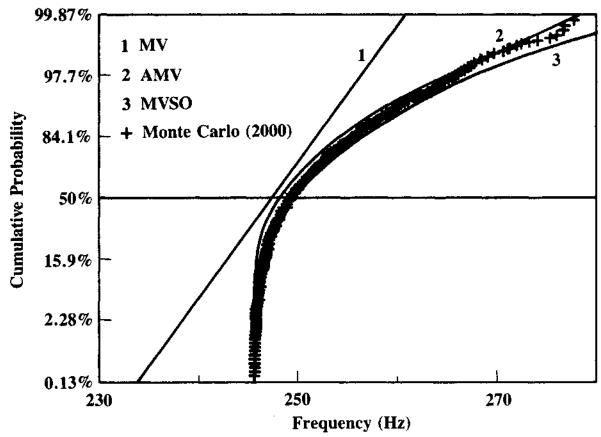

Fig. 8 MVSO and modified AMV solutions for example 2 (case 2: three random angles).

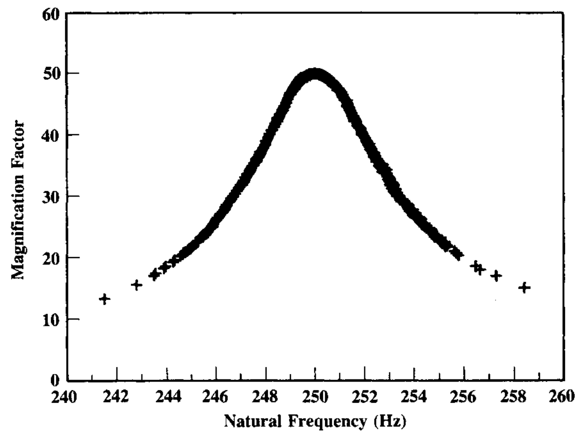

Fig. 9 Monte Carlo simulation of dynamic magnification factor (1000 samples).

Table 3 Variables for example 3

Variable	Mean	Standard deviation
f_n	250 Hz	2.5 Hz
f_e	250 Hz	0 Hz
ξ	0.01	0

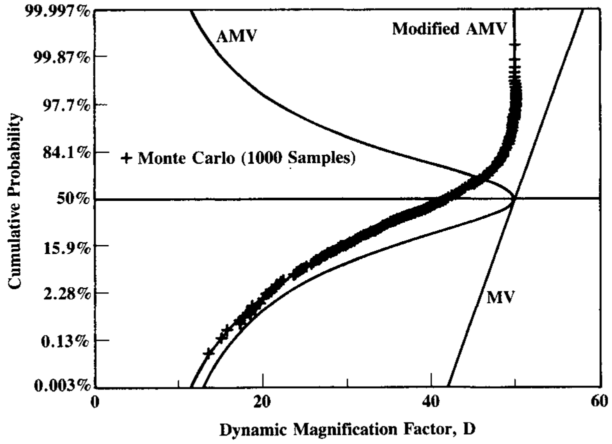

Fig. 10 Modified AMV solution for example 3.

Monte Carlo simulation was performed, and the empirical CDF, as shown in Fig. 6, agrees with the AMV solution. Thus, this simple case clearly shows how a concave performance function would produce a truncated distribution. The probability density function is shown in Fig. 7 to show how the distribution is skewed.

Case 2: Three Random Angles

In case 1, there exists only one independent random variable. In such a case, the generalized AMV solution is exact. In the present case, three random variables are assumed, as shown in Table 2. For multiple random variables, the AMV method offers approximate solutions.

For three random angles, it is difficult to obtain a closed form solution for f_n. Therefore, the NESSUS code was used to solve the problem numerically.

Following a similar solution procedure used in case 1, the MV and the AMV solutions are obtained, as shown in Fig. 8. The MV solution used four Z-function evaluations and the AMV solution used an additional 6 function evaluations. To confirm the results, a Monte Carlo simulation with 2000 samples was performed, with results plotted in Fig. 8.

The result of case 2 is different from case 1 because of the two additional random angles. However, the major characteristic remains the same; i.e., the distribution is truncated at the left tail.

It is noted that the error is relatively large at the central region (i.e., CDF ≈ 0.5). The reason is because f_n is strongly nonlinear at this region. The error can be reduced by an MVSO analysis. The MVSO solution, presented in Fig. 8, agrees with the simulation solution very well in the central region. However, the error appears to be significant at the right-tail region. This error would need to be improved by the AMV procedure or, perhaps, an AMV second-order procedure, which would be a natural extensioin of the first-order method. Because of the additions of the second-order terms, the AMV second-order method has the potential to be more accurate than the AMV first-order method. However, the required number of function evaluations will be greater. Future research will be needed to explore the AMV second-order method.

It is interesting to compare the MV solution with the AMV solution using the $\mu \pm 3\sigma$ design criteria. For the MV solution, because all three input random variables are normally distributed, f_{n1} is normally distributed; therefore, the $\mu \pm 3\sigma$ probability bounds are (0.0013, 0.9987). The corresponding frequencies are 234.0, 260.8 Hz. Keeping the same probability levels, the AMV-based bounds are 245.7, 278.3 Hz. Thus the MV solution significantly underestimates the bounds. Also, the actual probabilities corresponding to the $\mu \pm 3\sigma$ bounds are (0, 0.946), as opposed to (0.0013, 0.9987). This clearly demonstrates the importance of obtaining the entire CDF result.

Example 3: Dynamic Magnification Factor

This example was selected to represent a situation where the response CDF is truncated at the right tail.

Consider the first-mode dynamic characteristic of a turbine blade model. The dynamic magnification factor D is

$$D = \frac{1}{\sqrt{\left[1-\left(\frac{f_e}{f_n}\right)^2\right]^2 + \left[2\xi\left(\frac{f_e}{f_n}\right)\right]^2}} \tag{15}$$

where f_e is the exciting frequency, f_n is the natural frequency, and ξ is the damping factor. The variables are defined in Table 3, which shows that f_e and ξ are deterministic while f_n is random. As a worst-case analysis, it is assumed that f_e is equal to the mean value of f_n.

In this problem, D is a convex function of f_n. For each D there are two possible f_n values. Random samples of f_n were generated and used to calculate D using Eq. (15). Figure 9 shows the 1000 samples of D as a function of f_n.

Following the same procedure as described in the preceding examples, the Monte Carlo simulation, MV, AMV, and the modified AMV solutions are shown in Fig. 10. This problem demonstrates how a convex performance function results in a truncated distribution and how this truncated distribution can be identified quickly based on the generalized AMV concept.

Summary

The CDF analysis provides a more useful reliability design tool than the conventional mean-based second-moment analysis. This paper presented a generalized AMV method, which is capable of performing a CDF analysis in a robust and highly efficient manner. Several examples were used to demonstrate the methodology. The examples included monotonic as well as convex and concave performance functions. It was shown why a performance function CDF is truncated and how a truncated distribution can be identified and computed using the proposed procedure. The extension to the more general nonlinear functions has also been discussed in the paper. In summary, the generalized AMV method appears to be ideal to solve problems involving implicit, complex performance functions.

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